MANIFOLDS WITHOUT CONJUGATE POINTS AND THEIR FUNDAMENTAL GROUPS

SERGEI IVANOV AND VITALI KAPOVITCH

ABSTRACT. We show that in the fundamental groups of closed manifolds without conjugate points centralizers of all elements virtually split.

1. INTRODUCTION

It is a classical consequence of Rauch comparison that manifolds of nonpositive curvature have no conjugate points. While the converse need not hold even for closed manifolds [11], the following question remains unanswered in dimensions above 2.

Question 1.1. Does every closed Riemannian manifold without conjugate points admit a metric of nonpositive curvature?

While the answer to this is likely negative, it has been shown that fundamental groups of closed manifolds without conjugate points satisfy many properties which are known to hold for nonpositively curved manifolds. In particular, Croke and Schroeder [8] proved that if a closed manifold $\bar{M}$ admits an *analytic* metric without conjugate points then every abelian subgroup of $\pi_1(\bar{M})$ is straight (i.e. quasi-isometrically embedded in $\pi_1(\bar{M})$) and every solvable subgroup of $\pi_1(\bar{M})$ is virtually abelian. Both of these properties are known to hold for groups which act isometrically, properly and co-compactly on CAT(0)-spaces (we refer to such groups as CAT(0)-groups) or, more generally, for the semi-hyperbolic groups (which is a strictly bigger class of groups, see e.g. [1] or [5] for the definition). Our main result is that the following known property of CAT(0)-groups also holds for fundamental groups of closed manifolds without conjugate points.

Theorem A. *Let $\bar{M}$ be a closed manifold that admits a C^∞ Riemannian metric without conjugate points. Then for every nontrivial element $\gamma \in \pi_1(\bar{M})$, its centralizer $Z(\gamma) < \pi_1(\bar{M})$ virtually splits over γ.*

This means that there exists a finite index subgroup $G < Z(\gamma)$ which is isomorphic to a direct product $\mathbb{Z} \times G'$ so that γ corresponds to the generator of the $\mathbb{Z}$ factor.

Actually, we prove an equivalent but more convenient property of the centralizer (Corollary 4.3): the image of γ in $H_1(Z(\gamma)) = Z(\gamma)/[Z(\gamma), Z(\gamma)]$ is a non-torsion element.

The proof builds upon the already mentioned work of Croke and Schroeder [8]. The main result of [8] (i.e. the straightness of abelian subgroups) by itself does not imply the assertion of Theorem A but some intermediate results in [8] do. We remove the analyticity assumption from one of those results (see Proposition 3.1 below) and deduce Theorem A from it.

Remark 1.2. Kleiner (unpublished) and Lebedeva [14] found simpler proofs of the main result of [8] that work without analyticity. These proofs go via a different route that do not yield Theorem A.

Remark 1.3. Our proof requires C^∞ regularity of the metric or, more precisely, C^r regularity for some r depending on $n = \dim M$. This is needed in Lemma 3.3. We do not know whether Theorem A is true for C^r metrics for any fixed r.

Examples constructed in [6] show that Lemma 3.3 and an analogue of Proposition 3.1 (in a similar but not identical context) fail if $n \gg r$. However it might be possible that some features of metrics without conjugate points (e.g. the fact that displacement functions do not have critical points other than minima) can be used to squeeze more regularity out of the problem.

Remark 1.4. A straightforward modification of the proof shows that Theorem A holds for Finsler metrics as well, but we do not bother the reader with this generalization.

Organization of the paper. In Section 2 we provide necessary background material on the properties of universal covers of closed manifolds without conjugate points. In Section 3 we prove the key technical result Proposition 3.1 which allows us to remove the analyticity assumption from the arguments in [8]. In Section 4 we prove Theorem A. In Section 5 we construct new examples of manifolds that can be shown not to admit metrics without conjugate points by Theorem A but not by previously known results. In Section 6 we prove that fundamental groups of closed manifolds without conjugate points have solvable word and conjugacy problems (Theorem 6.1). In Section 7 we consider an analogue of Question 1.1 for metrics without *focal* points and prove that the answer is affirmative in dimension 3 (Theorem 7.1).

Lastly, in Section 8 we discuss a number of open problems concerning manifolds without conjugate points.

2. Preliminaries

In this section we collect the necessary preliminaries, borrowed mainly from [8].

Let $\bar{M}$ be a compact Riemannian manifold without conjugate points and M its universal cover. We represent the fundamental group $\pi_1(\bar{M})$ as the group Γ of deck transformations of M. This group acts on M by isometries and $\bar{M} = M/\Gamma$. We fix the notation $\bar{M}$, M and Γ for the rest of the paper.

Since the metric has no conjugate points, every geodesic in M is minimizing, $\exp_x \colon T_x M \to M$ is a diffeomorphism for every $x \in M$, and the Riemannian

distance function $d\colon M \times M \to \mathbb{R}_+$ is smooth outside the diagonal. All geodesics throughout the paper are parametrized by arc length.

Fix a nontrivial element $\gamma \in \Gamma$. The *displacement function* $d_\gamma\colon M \to \mathbb{R}_+$ is defined by

$$d_\gamma(x) = d(x, \gamma x), \qquad x \in M.$$

A complete geodesic $c\colon \mathbb{R} \to M$ is called an *axis* of γ if γ translates c forward along itself, i.e., there is a constant $L > 0$ such that $\gamma c(t) = c(t + L)$ for all $t \in \mathbb{R}$. Note that if c is an axis of γ, then the reverse geodesic $t \mapsto c(-t)$ is an axis of γ^{-1}. Let $A_\gamma \subset M$ denote the union of all axes of γ.

The following lemma summarizes several results from [8] that we will need in what follows.

Lemma 2.1. *In the above notation, the following holds.*

(1) *The function d_γ assumes a positive minimum, $\min d_\gamma$. The set of points $x \in M$ where $d_\gamma(x) = \min d_\gamma$, is equal to A_γ.*

(2) *The isometry γ translates all its axes by the same amount, namely $\min d_\gamma$. That is, if c is an axis of γ then $\gamma(c(t)) = c(t + \min d_\gamma)$ for all $t \in \mathbb{R}$.*

(3) $\min d_{\gamma^m} = m \cdot \min d_\gamma$ *for every integer $m \geq 1$.*

(4) A_γ *is equal to the set of critical points of d_γ. In particular d_γ has no critical points outside its minimum set.*

Proof. See [8, Lemma 2.1] and remarks there. □

The following lemma is proved in [8, Lemma 2.4], however it is not made very clear there that the proof does not depend on analyticity of the metric. Therefore we include a proof (essentially the same one) here.

Lemma 2.2. *The set A_γ is connected.*

Proof. Consider the centralizer $Z(\gamma) \subset \Gamma$ of γ. It is easy to see that the action of $Z(\gamma)$ on M preserves d_γ, hence d_γ induces a well defined function $\bar{d}_\gamma$ on $M/Z(\gamma)$. Furthermore $\bar{d}_\gamma$ is proper (see [8, Lemma 2.2]) and by Lemma 2.1(4) it has no critical point outsize its minimum set $\bar{A}_\gamma$. Therefore by Morse theory every sublevel set

$$\bar{U}_\varepsilon = \{x \in M/Z(\gamma) : \bar{d}_\gamma(x) \leq \min \bar{d}_\gamma + \varepsilon\}, \qquad \varepsilon > 0,$$

is a strong deformation retract of $M/Z(\gamma)$. Since the projection $\pi\colon M \to M/Z(\gamma)$ is a covering map, it follows that the set

$$U_\varepsilon := \pi^{-1}(\bar{U}_\varepsilon) = \{x \in M : d_\gamma(x) \leq \min d_\gamma + \varepsilon\}$$

is a strong deformation retract of M. Hence U_ε is connected. Since $A_\gamma = \bigcap_{\varepsilon > 0} U_\varepsilon$, it follows that A_γ is connected as well. □

Remark 2.3. The proof of Lemma 2.2 implies that $Z(\gamma)$ is finitely generated and finitely presented since it shows that $M/Z(\gamma)$ is homotopy equivalent to $\bar{U}_\varepsilon$ which is compact manifold with boundary.

The *Busemann function* of a (minimizing) geodesic $c\colon \mathbb{R} \to M$ is a function $b_c\colon M \to \mathbb{R}$ defined by

$$b_c(x) = \lim_{t \to +\infty} d(x, c(t)) - t.$$

The triangle inequality implies that the function $t \mapsto d(x, c(t)) - t$ is non-increasing, hence $b_c(x)$ is well defined and $b_c(x) \leq d(x, c(t)) - t$ for all $t \in \mathbb{R}$. Note that $b_c(c(t)) = -t$ for all $t \in \mathbb{R}$.

Clearly Busemann functions are 1-Lipschitz. Busemann functions are naturally translated by isometries, namely if $\alpha \colon M \to M$ is an isometry, then $b_c(x) = b_{\alpha c}(\alpha x)$ for all $x \in M$. Changing the origin of a geodesic adds a constant to its Busemann function, namely if $c_1(t) = c(t + L)$ where L is a constant, then $b_{c_1}(x) = b_c(x) + L$.

Lemma 2.4. *Let c and c_1 be axes of γ. Then*
  (1) $b_c(\gamma x) = b_c(x) - \min d_\gamma$ *for all $x \in M$.*
  (2) b_c *decays at unit rate along c_1, that is $b_c(c_1(t + t_1)) = b_c(c_1(t)) - t_1$ for all $t, t_1 \in \mathbb{R}$.*

Proof. Let $L = \min d_\gamma$. Then $b_c(\gamma x) = b_{\gamma^{-1}c}(x) = b_c(x) - L$ since $\gamma^{-1}c(t) = c(t - L)$ by Lemma 2.1(2). This proves the first assertion. It implies that

$$b_c(c_1(t + mL)) = b_c(\gamma^m c_1(t)) = b_c(c_1(t)) - mL$$

for all $m \in \mathbb{Z}$. Since b_c is 1-Lipschitz, the second assertion follows. $\square$

3. Busemann functions of axes of isometries

The goal of this section in to prove the following.

Proposition 3.1. *Let $\gamma \in \Gamma \setminus \{e\}$, and let c and c_1 be axes of γ. Then $b_c - b_{c_1}$ is constant on M.*

This proposition was proved in the analytic case in [8, Proposition 2.5]. The analyticity assumption is used in [8] to ensure some regularity of the set A_γ, namely it implies that this set is *locally rectifiably path connected* (see [8] for the definition). This property and local analysis of Busemann functions on A_γ yields the result. We do not know if A_γ is locally rectifiably path connected if the metric is only C^∞. We work around this issue by more delicate analysis of the behavior of Busemann functions in a neighborhood of A_γ (Lemma 3.2).

For a geodesic c in M, denote by b_c^- the Busemann function of the reverse geodesic $t \mapsto c(-t)$ and let $b_c^0 = b_c + b_c^-$. The definition of a Busemann function and the triangle inequality imply that $b_c^0(x) \geq 0$ for all $x \in M$.

Lemma 3.2. *Let c be an axis of γ and $L = \min d_\gamma$. Define $f(x) = d_\gamma(x) - L$ for $x \in M$. Then there exists $C > 0$ such that*

$$(3.1) \qquad |b_c^0(x) - b_c^0(y)| \leq C\big(d(x,y)^2 + f(x) + f(y)\big)$$

for all $x, y \in M$.

Proof. Note that $|b_c^0(x) - b_c^0(y)| \leq 2d(x, y)$ since Busemann functions are 1-Lipschitz. Therefore we may assume that $d(x, y)$ is small, more precisely $d(x, y) \leq L/2$. Indeed, if $d(x, y) > L/2$ then $|b_c^0(x) - b_c^0(y)| \leq 2d(x, y) \leq 4L^{-1}d(x, y)^2$, so (3.1) is satisfied for any $C \geq 4L^{-1}$. We may also assume that $f(x) \leq 1$ and $f(y) \leq 1$, otherwise

$$|b_c^0(x) - b_c^0(y)| \leq 2d(x, y) \leq L \leq L(f(x) + f(y)),$$

so (3.1) is satisfied for any $C \geq L$.

Let $x_+ = \gamma x$ and $x_- = \gamma^{-1}x$. Then

$$d(x, x_+) = d(x, x_-) = d_\gamma(x) = L + f(x).$$

By Lemma 2.4(1), we have $b_c(x) = b_c(x_+) + L$ and $b_c^-(x) = b_c^-(x_-) + L$. Since Busemann functions are 1-Lipschitz, it follows that

$$b_c(y) - b_c(x) = b_c(y) - b_c(x_+) - L \leq d(y, x_+) - L$$

and
$$b_c^-(y) - b_c^-(x) = b_c(y) - b_c^-(x_-) - L \le d(y, x_-) - L.$$
Summing these two inequalities yields

(3.2) $$\qquad b_c^0(y) - b_c^0(x) \le d(y, x_+) + d(y, x_-) - 2L.$$

Our plan is to estimate the right-hand side of this inequality.

For $p, q \in M$, denote by $\overrightarrow{pq}$ the initial velocity vector of the unique geodesic connecting p to q. That is, $\overrightarrow{pq}$ is the unit vector in $T_p M$ positively proportional to $\exp_p^{-1}(q)$. Let $v_+ = \overrightarrow{xx_+}$, $v_- = \overrightarrow{xx_-}$ and $z \in M$ be any point such that $d(x, z) \le L/2$. Then, by the first variation formula,

$$d(z, x_\pm) \le d(x, x_\pm) - \langle \overrightarrow{xz}, v_\pm \rangle \cdot d(x, z) + C_1 d(x, z)^2$$
$$= L + f(x) - \langle \overrightarrow{xz}, v_\pm \rangle \cdot d(x, z) + C_1 d(x, z)^2$$

and therefore

(3.3) $$\quad d(z, x_+) + d(z, x_-) \le 2L + 2f(x) - \langle \overrightarrow{xz}, v_+ + v_- \rangle \cdot d(x, z) + 2C_1 d(x, z)^2$$

where $C_1 > 0$ is a constant depending only on M and L. (This constant is just an upper bound for the second derivative of the distance to $x_\pm$ on the geodesic segment $[x, z]$. It is uniform in x because M admits a co-compact action by isometries and $L \le d(x, x_\pm) \le L + 1$.)

For $z = y$, (3.2) and (3.3) imply that

(3.4)
$$b_c^0(y) - b_c^0(x) \le 2f(x) - \langle \overrightarrow{xy}, v_+ + v_- \rangle \cdot d(x, y) + 2C_1 d(x, y)^2$$
$$\le 2f(x) + \sigma \cdot d(x, y) + 2C_1 d(x, y)^2.$$

where $\sigma = |v_+ + v_-|$.

It remains to estimate σ. Consider a point $z = \exp_x(t_0 v)$ where $v \in T_x M$ is the unit vector positively proportional to $v_+ + v_-$ (or any unit vector if $v_+ + v_- = 0$) and $t_0 = \sqrt{f(x)/C_1}$. We may assume that C_1 is sufficiently large so that $t_0 \le L/2$ (recall that $f(x) \le 1$). For the so defined z, (3.3) takes the form

$$d(z, x_+) + d(z, x_-) \le 2L + 2f(x) - \sigma t_0 + 2C_1 t_0^2.$$

On the other hand,

$$d(z, x_+) + d(z, x_-) \ge d(x_+, x_-) = d_{\gamma^2}(x_-) \ge \min d_{\gamma^2} = 2L$$

by the triangle inequality and Lemma 2.1(3) (recall that $x_+ = \gamma^2 x_-$). Hence

$$0 \le 2f(x) - \sigma t_0 + 2C_1 t_0^2 = -\sigma \sqrt{f(x)/C_1} + 4f(x),$$

or, equivalently, $\sigma \le 4\sqrt{C_1 f(x)}$. Therefore

$$\sigma \cdot d(x, y) \le 4\sqrt{C_1 f(x) d(x, y)^2} \le 2f(x) + 2C_1 d(x, y)^2.$$

Plugging this into (3.4) yields

$$b_c^0(y) - b_c^0(x) \le 4f(x) + 4C_1 d(x, y)^2$$

and, by switching the roles of x and y,

$$b_c^0(x) - b_c^0(y) \le 4f(y) + 4C_1 d(x, y)^2.$$

Therefore (3.1) is satisfied for $C \ge \max\{4C_1, 4\}$. $\qquad \square$

The following lemma is purely analytic and local. It does not use the assumption that M is co-compact and free of conjugate points.

Lemma 3.3. *Let M be any Riemannian manifold, $f \in C^\infty(M)$ and $g \colon M \to \mathbb{R}$ a continuous function such that*

$$(3.5) \qquad |g(x) - g(y)| \leq C\big(d(x,y)^2 + |f(x)| + |f(y)|\big)$$

for some constant $C > 0$ and all $x, y \in M$. Let $X = f^{-1}(0)$. Then the set $g(X) \subset \mathbb{R}$ has zero Lebesgue measure.

Proof. We argue by induction in $n = \dim M$. The case $n = 0$ is trivial. Assume that $n \geq 1$ and the lemma holds for all $(n-1)$-dimensional manifolds.

Since M can be covered by countably many coordinate neighborhoods, we may assume that M is an open ball in $\mathbb{R}^n$ and f extends to a smooth function on a compact set. Let Z be the set of points in X where all derivatives of f vanish, and let $Y = X \setminus Z$.

The set Y is contained in a countable union of $(n-1)$-dimensional smooth submanifolds. Indeed, for each multi-index $I = (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$, $k \geq 0$, consider the partial derivative $f_I = \frac{\partial^k f}{\partial x_1 \ldots \partial x_k}$. Let Σ_I be the set of points in M where the function f_I vanish but its first derivative does not. Since $f_I \in C^\infty$, Σ_I is a smooth $(n-1)$-dimensional submanifold, and Y is contained in the union of all such submanifolds.

Applying the induction hypothesis to Σ_I in place of M yields that $g(X \cap \Sigma_I)$ is a set of measure zero for every multi-index I. Therefore $g(Y)$ has measure zero.

It remains to handle the set $g(Z)$. Since f and all its derivatives up to the order $2n$ vanish on Z and are bounded on M, there is a constant $C_1 > 0$ such that

$$(3.6) \qquad |f(x)| = |f(x) - f(z)| \leq C_1 |x - z|^{2n}$$

for all $x \in M$ and $z \in Z$. We are going to show that

$$(3.7) \qquad |g(z) - g(z')| \leq C_2 |z - z'|^{n+1}$$

for some constant $C_2 > 0$ and all $z, z' \in Z$ such that $|z - z'| \leq 1$. Let $\delta = |z - z'|$ and pick a positive integer N such that $\delta^{1-n} \leq N \leq 2\delta^{1-n}$. Divide the segment $[z_1, z_2]$ into N segments $[x_{i-1}, x_i]$, $i = 1, \ldots, N$, of equal lengths. Note that

$$|x_i - x_{i-1}| = \delta/N \leq \delta^n$$

by the choice of N, and

$$|f(x_i)| + |f(x_{i-1})| \leq C_1 |x_i - z|^{2n} + C_1 |x_{i-1} - z|^{2n} \leq 2C_1 \delta^{2n}$$

by (3.6). Substituting x_i and x_{i-1} for x and y in (3.5) yields that

$$|g(x_i) - g(x_{i-1})| \leq C\big(|x_i - x_{i-1}|^2 + |f(x_i)| + |f(x_{i-1})|\big) \leq C(1 + 2C_1)\delta^{2n}.$$

Hence

$$|g(z) - g(z')| = \Big|\sum g(x_i) - g(x_{i-1})\Big| \leq C(1 + 2C_1)\delta^{2n} N \leq 2C(1 + 2C_1)\delta^{n+1}.$$

Thus (3.7) holds for $C_2 = 2C(1 + 2C_1)$.

The inequality (3.7) implies that

$$\dim_H(g(Z)) \leq \frac{\dim_H(Z)}{n+1} \leq \frac{\dim_H(M)}{n+1} = \frac{n}{n+1} < 1$$

where $\dim_H$ denotes the Hausdorff dimension. Hence the one-dimensional measure of $g(Z)$ is zero and the lemma follows. $\qquad\square$

Corollary 3.4. *Let c be an axis of γ. Then $b_c^0 = 0$ on A_γ.*

Proof. Let $L = \min d_\gamma$, then Lemma 3.2 asserts that the functions $f = d_\gamma - L$ and $g = b_c^0$ satisfy the assumptions of Lemma 3.3. Since $A_\gamma = f^{-1}(0)$, Lemma 3.3 implies that $b_c^0(A_\gamma)$ is a set of measure zero in $\mathbb{R}$. By Lemma 2.2, A_γ and hence $b_c^0(A_\gamma)$ are connected. Since the set $b_c^0(A_\gamma)$ is connected and has measure zero, it is a single point on the real line. This means that b_c^0 is constant on A_γ. Since $c(0) \in A_\gamma$ and $b_c^0(c(0)) = 0$, this constant is zero. $\qquad\square$

Proof of Proposition 3.1. Let c and c_1 be axes of γ. Changing the origin of c_1 we may assume that $b_c(c_1(0)) = 0$. Then we have to prove that $b_c = b_{c_1}$.

First we show that the relation $b_c(c_1(0)) = 0$ is symmetric, i.e. it implies that $b_{c_1}(c(0)) = 0$. Since $c_1(0) \in A_\gamma$, we have $b_c^0(c_1(0)) = 0$ by Corollary 3.4, hence $b_c^-(c_1(0)) = 0$. This means that

$$d(c_1(0), c(-t)) = t + \varepsilon(t)$$

where $\varepsilon(t) \to 0$ as $t \to +\infty$. Since $b_{c_1}(c(-t)) \leq d(c(-t), c_1(0))$, it follows that $b_{c_1}(c(-t)) \leq t + \varepsilon(t)$. This and Lemma 2.4(2) imply that

$$b_{c_1}(c(0)) = b_{c_1}(c(-t)) - t \leq \varepsilon(t)$$

for all $t > 0$. Therefore $b_{c_1}(c(0)) \leq 0$. Similarly, $b_{c_1}^-(c(0)) \leq 0$. Both inequalities must turn to equalities because $b_{c_1} + b_{c_1}^- \geq 0$. Thus $b_{c_1}(c(0)) = 0$ as claimed.

Let $x \in M$ and $\varepsilon > 0$. By the definition of Busemann function, there exists $t_1 \in \mathbb{R}$ such that

$$d(x, c_1(t_1)) < b_{c_1}(x) + t_1 + \varepsilon.$$

Since $b_c(c_1(0)) = 0$, Lemma 2.4(2) implies that $b_c(c_1(t_1)) = -t_1$, hence

$$d(c_1(t_1), c(t_0)) < t_0 - t_1 + \varepsilon$$

for a sufficiently large $t_0 \in \mathbb{R}$. Therefore

$$d(x, c(t_0)) \leq d(x, c_1(t_1)) + d(c_1(t_1), c(t_0)) < b_{c_1}(x) + t_0 + 2\varepsilon$$

and hence $b_c(x) \leq b_{c_1}(x)$ since ε is arbitrary.

Swapping c and c_1 in this argument yields that $b_{c_1}(x) \leq b_c(x)$. Thus $b_c = b_{c_1}$ and the proposition follows. $\qquad\square$

4. Virtual splitting of centralizer

Lemma 4.1. *Let $\alpha \in Z(\gamma)$ and c be an axis of γ. Then $b_c(\alpha x) - b_c(x)$ does not depend on $x \in M$.*

Proof. This is [8, Corollary 2.6] without the analyticity assumption. Observe that $\alpha^{-1}c$ is an axis of γ, indeed, if $L = \min d_\gamma$ then

$$\gamma\alpha^{-1}c(t) = \alpha^{-1}\gamma c(t) = \alpha^{-1}c(t + L).$$

Therefore $b_{\alpha^{-1}c} - b_c$ is constant by Proposition 3.1. Since $b_{\alpha^{-1}c}(x) = b_c(\alpha x)$, the lemma follows. $\qquad\square$

Corollary 4.2. *There exist a homomorphism $h\colon Z(\gamma) \to \mathbb{R}$ such that $h(\gamma) \neq 0$.*

Proof. Let c be an axis of γ. For $\alpha \in Z(\gamma)$, define

$$h(\alpha) = b_c(\alpha x) - b_c(x)$$

where $x \in M$ is an arbitrary point. By Lemma 4.1, $h(\alpha)$ is well defined. For $\alpha, \beta \in Z(\gamma)$ we have

$$h(\alpha\beta) = b_c(\alpha\beta x) - b_c(x) = b_c(\alpha\beta x) - b_c(\beta x) + b_c(\beta x) - b_c(x) = h(\alpha) + h(\beta).$$

Thus h is a homomorphism. By Lemma 2.4(1), $h(\gamma) = -\min d_\gamma \neq 0$. $\qquad\square$

Recall that $H_1(Z(\gamma)) = Z(\gamma)/[Z(\gamma), Z(\gamma)]$. We denote by π the projection from $Z(\gamma)$ to $H_1(Z(\gamma))$.

Corollary 4.3. *$\pi(\gamma)$ is non-torsion (i.e. has infinite order) in $H_1(Z(\gamma))$.*

Proof. Since $\mathbb{R}$ is commutative, the homomorphism h from Corollary 4.2 factors as $h = \bar{h} \circ \pi$ where $\bar{h}$ is a homomorphism from $H_1(Z(\gamma))$ to $\mathbb{R}$. Since $\bar{h}(\pi(\gamma)) \neq 0$, $\pi(\gamma)$ has infinite order. $\qquad\square$

Proof of Theorem A. Recall that by Remark 2.3 $Z(\gamma)$ is finitely generated and hence so is $H_1(Z(\gamma))$. By Corollary 4.3 and the classification of finitely generated abelian groups, $\pi(\gamma)$ belongs to a finite index subgroup $H < H_1(Z(\gamma))$ which is isomorphic to $\mathbb{Z}^k$ so that $\pi(\gamma)$ is mapped to $(1, 0, \ldots, 0) \in \mathbb{Z}^k$ by this isomorphism. Let $p \colon H \cong \mathbb{Z}^k \to \mathbb{Z}$ be first coordinate projection. Then $G := \pi^{-1}(H)$ is a finite index subgroup of $Z(\gamma)$ and $\phi = p \circ \pi \colon G \to \mathbb{Z}$ is a homomorphism sending γ to $1 \in \mathbb{Z}$. The existence of such a homomorphism and the fact that γ belongs to the center of G imply that $G \cong \mathbb{Z} \times G'$ where the $\mathbb{Z}$ factor corresponds to the subgroup generated by γ and $G' = \ker \phi$. $\qquad\square$

5. Examples

In this section we will give some new examples of manifolds that can be shown not to admit metrics without conjugate points using Theorem A.

Example 5.1. Let S_g^2 be a closed orientable surface of genus $g > 1$ equipped with any Riemannian metric. Let $\bar{M}^3 = T^1 S_g^2$ be the unit tangent bundle to S_g^2. Then $\bar{M}^3$ does not admit a metric without conjugate points.

Indeed, since the Euler class of S_g^2 is not equal to zero, $\pi_1(\bar{M})$ is a nontrivial central extension $1 \to \mathbb{Z} \to \pi_1(\bar{M}) \to \pi_1(S_g^2) \to 1$. By Theorem A it does not admit a metric without conjugate points. On the other hand, $\pi_1(\bar{M})$ is semi-hyperbolic [1] and therefore it satisfies the previously known restrictions on fundamental groups of manifolds without conjugate points proved in [8], namely, every abelian subgroup of $\pi_1(\bar{M})$ is straight and every solvable subgroup of $\pi_1(\bar{M})$ is virtually abelian.

Recall that the key step in the proof of Theorem A is Proposition 3.1. It was proved in [8] under the assumption that A_γ is locally rectifiably path connected. As was discussed earlier this can be guaranteed if the metric is real analytic. Another condition that obviously ensures it is if γ belongs to the center of $\pi_1(\bar{M})$. Indeed, if $\gamma \in Z(\pi_1(\bar{M}))$ then d_γ descends to a well-defined function on M and hence it is bounded above and attains its maximum. By Lemma 2.1(4) this means that $\max d_\gamma = \min d_\gamma$, i.e. d_γ is constant. That means that $A_\gamma = M$ which is obviously rectifiably path connected. Therefore one can prove that $\bar{M}$ admits no metric without conjugate points without using Proposition 3.1.

Next we will give an example which has no finite index subgroups with nontrivial center and requires the full strength of Proposition 3.1 and Theorem A.

Example 5.2. First let us recall some basic facts about diffeomorphism groups of surfaces. Let F be a closed orientable surface of genus > 1. Then the mapping class group $\mathrm{Mod}(F)$ is defined as $\pi_0(\mathrm{Diff}(F))$. It can be alternatively described as $\mathrm{Diff}(F)/\mathrm{Diff}_0(F))$ where $\mathrm{Diff}_0(F)$ is the identity component of $\mathrm{Diff}(F)$. By a classical result of Dehn and Nielsen $\mathrm{Mod}(F)$ is isomorphic to the group

$\operatorname{Out}(\pi_1(F)) = \operatorname{Aut}(\pi_1(F))/\operatorname{Inn}(\pi_1(F))$ of outer automorphisms of $\pi_1(F)$. For a once-punctured surface $F \setminus \{pt\}$, we have $\operatorname{Mod}(F \setminus \{pt\}) \cong \operatorname{Aut}(\pi_1(F))$.

It is well known that $\operatorname{Diff}_0(F)$ is contractible [9].

Lemma 5.3. *Let S_g^2 be a closed orientable surface of genus $g > 1$, $\bar{M}^3 = T^1 S_g$ and $F = S_g \# S_g$. Then there exists a monomorphism $\rho \colon \pi_1(\bar{M}) \to \operatorname{Mod}(F)$ which admits a lift to a monomorphism $\bar{\rho} \colon \pi_1(\bar{M}) \to \operatorname{Mod}(F \setminus \{pt\})$.*

Proof. Pick $p \in S_g$ and $v \in T_p^1 S_g$. Let $\operatorname{Diff}(S_g, v)$ be the subgroup of $\operatorname{Diff}(S_g)$ fixing v. Consider the fibration $\operatorname{Diff}(S_g, v) \to \operatorname{Diff}(S_g) \overset{ev}{\to} \bar{M}^3$ where the last map is just the evaluation map on v. Look at the long exact homotopy sequence of this fibration

$$\ldots \to \pi_1(\operatorname{Diff}(S_g)) \to \pi_1(\bar{M}^3) \overset{i}{\to} \pi_0(\operatorname{Diff}(S_g, v)) \to \pi_0(\operatorname{Diff}(S_g))$$

As mentioned earlier $\operatorname{Diff}_0(S_g)$ is contractible; therefore $\pi_1(\operatorname{Diff}(S_g)) = 1$ and hence $\pi_1(\bar{M}^3)$ injects as a subgroup into $\pi_0(\operatorname{Diff}(S_g, v))$. Next choose an $\varepsilon > 0$ much smaller than the injectivity radius of S_g and let $S^+ = S_g \setminus B_\varepsilon(p)$. Then $\operatorname{Diff}(S_g, v)$ is naturally isomorphic to $\operatorname{Diff}(S^+, \partial S^+)$, the group of diffeomorphisms of S^+ fixing the boundary ∂S^+ pointwise. Let $F = S^+ \cup_{\partial S^+} S^-$ where S^- is another copy of S^+ (i.e. F is the double of S^+ along its boundary). Then $\operatorname{Diff}(S^+, \partial S^+)$ naturally embeds into $\operatorname{Diff}(F)$ by extending the diffeomorphisms to S^- by identity. This induces a map $j \colon \pi_0(\operatorname{Diff}(S^+, \partial S^+)) \to \pi_0(\operatorname{Diff}(F))$ which is also injective. Then $\rho = j \circ i \colon \pi_1(\bar{M}) \to \operatorname{Mod}(F)$ is injective.

The above construction (originally due to Mess [16]) is borrowed from [13, Section 4.1], see also [2].

Next observe that if instead of gluing S^- to S^+ we glue $S^- \setminus \{pt\}$ the same argument gives an embedding $\pi_1(\bar{M}) \to \operatorname{Mod}(F \setminus \{pt\})$ so that the above map $\rho \colon \pi_1(\bar{M}) \to \operatorname{Out}(\pi_1(F))$ lifts to a map $\bar{\rho} \colon \pi_1(\bar{M}) \to \operatorname{Mod}(F \setminus \{pt\})$ and both of these are injections. $\qquad\square$

Let $B\operatorname{Diff}(F) = E/\operatorname{Diff}(F)$ be the classifying space of $\operatorname{Diff}(F)$ (here E is contractible). Then the projection map $p \colon E \to B\operatorname{Diff}(F)$ factors through the map

$$E \to E/\operatorname{Diff}_0(F) \overset{\bar{p}}{\to} E/\operatorname{Diff}(F) = B\operatorname{Diff}(F)$$

where $\bar{p}$ is just the quotient by $\operatorname{Diff}(F)/\operatorname{Diff}_0(F)) = \operatorname{Mod}(F)$. Since $\operatorname{Diff}_0(F)$ is contractible then so is $E/\operatorname{Diff}_0(F)$ and therefore $B\operatorname{Diff}(F)$ can be identified with $B\operatorname{Mod}(F)$ which we will do from now on. Therefore, for any CW complex B any map $f \colon B \to B\operatorname{Mod}(F)$ (which up to homotopy is determined by the map on the fundamental groups) gives rise to a pullback bundle $\operatorname{Diff}(F) \to X \to B$ and its associated bundle $F \to N \to B$ coming from the natural action of $\operatorname{Diff}(F)$ on F.

Now consider the map $\bar{M} \to B\operatorname{Mod}(F)$ corresponding to the monomorphism $\rho \colon \pi_1(\bar{M}) \to \operatorname{Mod}(F)$ from Lemma 5.3 and pull back the universal F bundle. This gives us a 5-manifold N fibering over $\bar{M}$ with fiber F. Clearly N is aspherical and its fundamental groups fits into a short exact sequence

$$1 \to \pi_1(F) \to \pi_1(N) \to \pi_1(\bar{M}) \to 1.$$

Since ρ can be lifted to $\bar{\rho} \colon \pi_1(\bar{M}) \to \operatorname{Mod}(F \setminus \{pt\}) \cong \operatorname{Aut}(\pi_1(F))$, this exact sequence splits, i.e. $\pi_1(N)$ is isomorphic to the semidirect product $\pi_1(F) \rtimes_{\bar{\rho}} \pi_1(\bar{M})$.

Since $\bar{\rho}$ is injective, $\pi_1(N)$ has trivial center. On the other hand it contains $\pi_1(\bar{M})$ as a subgroup and therefore it has an element whose centralizer does not

virtually split. Therefore, N does not admit a metric without conjugate points by Theorem A. Lastly note that since ρ is injective, the whole group $\pi_1(N)$ injects into $\mathrm{Aut}(\pi_1(F))$ via the conjugation action. Also recall that $\mathrm{Aut}(\pi_1(F)) \cong \mathrm{Mod}(F \setminus \{pt\})$. It is well known that mapping class groups of hyperbolizable surfaces satisfy the property that all their abelian subgroups are straight (see [10] and [12, Lemma 8.7]) and all their solvable subgroups are virtually abelian [3]. Of course the same is therefore true for all their subgroups and hence it is true for $\pi_1(N)$.

6. Solvability of word and conjugacy problems

It is very well known that fundamental groups of manifolds of nonpositive curvature have solvable word and conjugacy problems. We observe that the same is true for fundamental groups of closed manifolds without conjugate points.

Theorem 6.1. *Let $\bar{M}$ be a closed manifold with a Riemannian metric without conjugate points. Then $\pi_1(\bar{M})$ has solvable word and conjugacy problems.*

Proof. The proof is a straightforward combination of known results. The key is the following Lemma.

Lemma 6.2. *Let $c_0, c_1 \colon S^1 \to \bar{M}$ be two freely homotopic rectifiable loops with length $L(c_i) < C$ for $i = 0, 1$. Then there exists a free homotopy $F \colon S^1 \times [0,1] \to \bar{M}$ from c_0 to c_1 through loops c_t such that $L(c_t) < C$ for any $t \in [0,1]$.*

Proof. Lifting the homotopy to the universal cover M we get a homotopy between two curves $\tilde{c}_0, \tilde{c}_1 \colon [0,1] \to M$. Since the metric has no conjugate points, it is easy to connect each $\tilde{c}_i$, $i = 1, 2$, to the (unique) geodesic segment $[\tilde{c}_i(0), \tilde{c}_i(1)]$ by a homotopy fixing the endpoints and not increasing lengths. (For example, consider a family of curves $\{\tilde{c}_{i,t}\}, t \in [0,1]$, where $\tilde{c}_{i,t}$ is the concatenation of the geodesic segment $[\tilde{c}_i(0), \tilde{c}_i(t)]$ and the arc $\tilde{c}_i|_{[t,1]}$ of $\tilde{c}_i$.) Thus we may assume that $\tilde{c}_0$ and $\tilde{c}_1$ are geodesics.

Let γ be the element of the deck transformation group such that $\gamma(\tilde{c}_0(0)) = \tilde{c}_0(1)$, then $\gamma(\tilde{c}_1(0)) = \tilde{c}_1(1)$. Thus by our assumption both $\tilde{c}_0(0)$ and $\tilde{c}_1(0)$ belong to the set $\{d_\gamma < C\}$. By Lemma 2.1(4) and the proof of Lemma 2.2 the set $\{d_\gamma < C\}$ is path connected and therefore we can find a path $\alpha \colon [0,1] \to \{d_\gamma < C\}$ such that $\alpha(0) = \tilde{c}_0(0)$ and $\alpha(1) = \tilde{c}_1(0)$. Connect $\alpha(s)$ to $\gamma(\alpha(s))$ by shortest geodesics for all $s \in [0,1]$. They are unique and vary continuously since M has no conjugate points. Projecting this family of geodesics to $\bar{M}$ produces a homotopy with desired properties. $\qquad\square$

We are indebted to Ilya Kapovich for providing the following argument that the property provided by Lemma 6.2 implies solvability of the word problem in $\pi_1(\bar{M})$. Since the argument is apparently well-known we only give a brief outline.

It is well-known that to prove solvability of the word problem in a finitely presented group it is enough to show that its Dehn function has a recursive upper bound (see for example [4, Theorem 1.5.8, p. 104]).

We claim that Lemma 6.2 yields an exponential upper bound on the Dehn function of $\pi_1(\bar{M})$.

Fix a finite presentation $\langle F|R \rangle$ of $\Gamma = \pi_1(\bar{M}, p)$ and a K-quasi-isometry between M and the Caley graph of Γ with respect to that presentation sending p to e.

Let $F \colon S^1 \times [0,1] \to \bar{M}$ be a homotopy between a constant loop c_0 at p and a loop c_1. Let $C = 2L(c_1)$. Since we are only interested in large scale estimates

we can assume that $C \geq 2 \operatorname{diam} \bar{M}$. By Lemma 6.2 we can change F to a free homotopy such that $L(c_t) \leq C$ for any t.

Consider a sufficiently fine subdivision of $[0,1]$ given by $0 = t_0 < t_1 < \ldots < t_N = 1$. For every $j = 0, \ldots N$ connect $c_{t_j}(0)$ to the fixed base point $p = c_0(0)$ by a shortest geodesic. For every j this produces a nullhomotopic loop c'_j based at p of length $\leq C + 2 \operatorname{diam}(\bar{M})$. Look at the corresponding curves in the Caley graph and the words they represent in the presentation $\langle F | R \rangle$. They all have lengths $\leq K(C + 2 \operatorname{diam}(\bar{M})) \leq 2KC$. The ball of radius R in the Caley graph has at most $\exp(C_2 \cdot R)$ vertices for some explicit constant C_2 depending on $|F|$. Therefore the loops c'_j produce at most $\exp(2C_2 KC)$ distinct words in the group. Therefore, if the original homotopy was "too long" we can cut out the parts between repeating words and reduce its length. This produces a homotopy in the Caley graph of area $\leq \exp(C_3 \cdot C)$ for some easily computable constant $C_3 = C_3(C_2, K, |F| + |R|)$. Therefore the Dehn function grows at most exponentially.

Lastly, by [5, section 1.11, p. 446] if $\pi_1(\bar{M})$ has a solvable word problem and satisfies the conclusion of Lemma 6.2, then it also has solvable conjugacy problem.
$\qquad\square$

Remark 6.3. One can also prove the solvability of the word problem in Γ as follows. Since $\bar{M}$ is compact we have that $|sec(M)| \leq K$ for some $K > 0$ which trivially implies that for any $p \in M$ the exponential map $\exp_p \colon T_pM \to M$ is $\exp(C_1 R)$ Lipschitz on the R-ball $B_R(0) \subset T_pM$ for some $C_1 > 0$ and any $R > 0$. Using that geodesics in M are globally distance minimizing it is not hard to show that $\exp_p^{-1}$ is also $\exp(C_2 R)$ Lipschitz which implies an exponential bound on the isoperimetric function in M and hence the Dehn function of Γ also grows at most exponentially.

It is well-known that Dehn functions of $CAT(0)$-groups grow at most quadratically. It is reasonable to expect that the same should be true for fundamental groups of closed manifolds without conjugate points but we have been unable to prove it.

7. 3-MANIFOLDS WITHOUT FOCAL POINTS

Recall that a Riemannian manifold M is said to have no focal points if every embedded shortest geodesic $c \colon (a, b) \to M$ has no focal points when viewed as a submanifold of M. The class of manifolds without focal points is contained in the class of manifolds without conjugate points and contains all non positively curved manifolds. It is therefore natural to wonder if closed manifolds without focal points always admit nonpositively curved metrics.

The main result of this section is the following theorem which answers this question affirmatively in dimension 3.

Theorem 7.1. *Let M be a closed 3-manifold. Then M admits a Riemannian metric without focal points if and only if it admits a Riemannian metric of nonpositive sectional curvature.*

One of the main technical tools in the proof is the following result due to O'Sullivan [17].

Theorem 7.2 (Flat torus theorem)**.** [17, Theorem 2] *Let M be a closed Riemannian manifold without focal points. Let $A \leq \pi_1(M)$ be a solvable subgroup. Then there exists a flat totally geodesically embedded space form $i \colon F^k \hookrightarrow M$ such that $i_* \colon \pi_1(F^k) \to \pi_1(M)$ is injective and A is a finite index subgroup in $i_*(\pi_1(F^k))$.*

We also need the following result from the same paper.

Proposition 7.3. [17, Proposition 5] *Let $\bar{M}$ be a closed manifold without focal points and γ a nontrivial element of the deck transformation group $\Gamma \cong \pi_1(\bar{M})$. Then*

(1) *The union of axes A_γ is convex and closed in the universal cover M.*
(2) *A_γ is isometric to $\mathbb{R} \times N$ where N is a smooth submanifold of $\bar{M}$, possibly with boundary. The action of γ preserves this splitting and acts by translations by $\min d_\gamma$ on the $\mathbb{R}$-factors.*

Proof of Theorem 7.1. To simplify the exposition we will only give the proof for orientable manifolds. The non-orientable case is treated similarly. Alternatively we can appeal to a result of M. Kapovich and Leeb [13, Corollary 2.5] that given a finite cover $M_1 \to M_2$ between closed 3-manifolds, M_1 admits a metric of nonpositive sectional curvature if and only if M_2 does.

Let $\bar{M}$ be a closed orientable 3-manifold without focal points. Since $\bar{M}$ is aspherical it is prime. By the geometrization it admits a geometric decomposition. By [8] or by Theorem 7.2, none of the geometric pieces are sol or nil. Also, if $\bar{M}$ is Seifert fibered then by Theorem A it is finitely covered by a product $S^1 \times S_g$ where S_g is a closed surface of genus ≥ 2. All such manifolds obviously admit nonpositively curved metrics and hence so does $\bar{M}$. Thus we may assume that $\bar{M}$ is not Seifert fibered. If its geometric decomposition contains at least one hyperbolic piece then $\bar{M}$ admits a metric of nonpositive curvature by [15]. Thus we may assume that $\bar{M}$ is orientable, aspherical, not Seifert fibered and its geometric decomposition has no nil or sol pieces. This means that $\bar{M}$ is a graph manifold and all its geometric pieces are modelled on $\mathbb{H}^2 \times \mathbb{R}$.

Let us consider a single geometric piece E of $\bar{M}$. It is Seifert fibered over a hyperbolizable 2-manifold with boundary. By Theorem 7.2 we can assume that all the boundary tori of E (which are incompressible in $\bar{M}$) are flat and totally geodesic in $\bar{M}$. Let $E' \to E$ be a finite normal cover which is topologically a product $S^1 \times \Sigma$ where Σ is a compact 2-manifold with boundary. Let γ be a nontrivial element of $\pi_1(\bar{M})$ corresponding to the S^1 factor of E'. Let $\pi \colon M \to \bar{M}$ be the universal cover of $\bar{M}$ and let $p \colon \tilde{E} \to E$ be the universal cover of E. Since the inclusion $E \hookrightarrow \bar{M}$ is π_1-injective we can think of $\tilde{E}$ as a subset of M.

Clearly, the preimages of the boundary tori of E in $\tilde{E}$ belong to A_γ. This easily implies that the whole $\tilde{E}$ is contained in A_γ by essentially the same argument as in Example 5.1. Indeed, since $\pi_1(E') \subset Z(\gamma)$ in $\pi_1(\bar{M})$, d_γ descends from $\tilde{E}$ to a well-defined function d'_γ on E' and since E' is compact this function attains its maximum and minimum there. The whole boundary of E' belongs to the set of minima of d'_γ and since d_γ has no critical points outside A_γ by Lemma 2.1(4) we get that d'_γ is constant on E' and d_γ is constant on $\tilde{E}$.

Therefore by Proposition 7.3(2) $\tilde{E}$ isometrically splits as a product. More precisely, $\tilde{E}$ is isometric to $\mathbb{R} \times N^2$ where N is the universal cover of Σ equipped with some Riemannian metric with totally geodesic boundary. The isometric action of $\pi_1(E') = \pi_1(S^1) \times \pi_1(\Sigma)$ on $\mathbb{R} \times N$ preserves this splitting, furthermore the action of the $\pi_1(S^1)$ factor preserves the $\mathbb{R}$-fibers of $\mathbb{R} \times N$ and the action of the $\pi_1(\Sigma)$ factor descends to the action on N by deck transformations. Thus the metric of N descends to a well defined Riemannian metric g_0 on Σ. Note that we *do not* claim that E' itself is isometric to a product.

Let us apply the uniformization procedure to the double of Σ. It produces a metric g_1 of constant negative curvature in the same conformal class as the original metric g_0. Since the uniformization procedure commutes with isometries the boundary of Σ remains totally geodesic. Also, the isometric action (by deck transformations) of $\pi_1(\Sigma)$ on N with respect to $\tilde{g}_0$ remains isometric with respect to $\tilde{g}_1$. Therefore the action of $\pi_1(E)$ on $\tilde{E}$ is isometric with respect not only to the original product metric $\mathbb{R} \times (N, \tilde{g}_0)$ but also with respect to the new metric $\mathbb{R} \times (N, \tilde{g}_1)$ which is nonpositively curved. Notice that the conformal change of g_0 to g_1 may change the lengths of the boundary circles of Σ but we can easily modify the metric g_1 to g_2 by rescaling and attaching tubular collars near the boundary so that the g_2 is still nonpositively curved and all the boundary circles have the same lengths as in g_0 and are still totally geodesic. Moreover we can can easily arrange that g_2 is a product near the boundary. This yields a metric on E which is nonpositively curved, ∂E is totally geodesic and isometric to the original boundary, and the metric near ∂E is flat. Doing it on all geometric pieces of $\bar{M}$ separately and gluing the resulting metrics together yields a metric of nonpositive curvature on $\bar{M}$. □

8. Open problems

The main open question concerning manifolds without conjugate points is Question 1.1 which we restate here.

Question 8.1. Does every closed Riemannian manifold without conjugate points admit a nonpositively curved metric? The same can be asked about manifolds without focal points.

While we suspect that this is likely false in general, it might be true in dimension 3 (cf. Theorem 7.1). The main problem is to understand which graph manifolds admit metrics without conjugate points. The corresponding question about nonpositively curved metrics is completely understood by work of Buyalo and Kobelski [7].

The simplest test case to understand is the following

Question 8.2. Let $\Sigma = T^2 \setminus D^2$. Let $\bar{M}_1 = \bar{M}_2 = \Sigma \times S^1$. Let $\bar{M} = \bar{M}_1 \cup_f \bar{M}_2$ where f is a self diffeomorphism of the boundary torus $T^2 = \partial \bar{M}_1$ whose action on $\mathbb{Z}^2 = \pi_1(T^2)$ is given by a matrix $A \in SL(2, \mathbb{Z})$ with $|\mathrm{trace}(A)| > 2$. Then it is easy to see that $\bar{M}$ does not admit a metric of nonpositive curvature (say, by Theorem 7.2 and Proposition 7.3). *Does $\bar{M}$ admit a metric without conjugate points?*

Question 8.3. One of the natural classes containing CAT(0)-groups is the class of semi-hyperbolic groups. As was remarked in the introduction it is known that in semi-hyperbolic groups all abelian subgroups are straight and all solvable subgroups are virtually abelian. It is therefore natural to pose the following weaker version of Question 1.1.

Is the fundamental group of a closed manifold without conjugate points semi-hyperbolic?

Note however, that Theorem A *does not* hold for general semi-hyperbolic groups with the fundamental groups of unit tangent bundles to surfaces of genus > 1 providing the simplest counterexamples.

Lastly, let us note that it is easy to see that if $\bar{M}$ has no focal points then $\pi_1(\bar{M})$ is semi-hyperbolic. Indeed, no focal points condition implies that if $c_1(t), c_2(t)$ are

geodesics in M with $c_1(0) = c_2(0)$ then $d(c_1(t), c_2(t))$ is monotone increasing for $t > 0$ [17, Proposition 2]. This trivially implies that the canonical bicombing of M by shortest geodesics satisfies the fellow traveller property and hence $\pi_1(\bar{M})$ is semi-hyperbolic.

Question 8.4. Let $\bar{M}$ be a closed manifold without conjugate points and $\gamma \in \pi_1(\bar{M})$. Is it true that $Z(\gamma)$ is straight in $\pi_1(\bar{M})$? This is easily seen to be true if $\bar{M}$ has no focal points because in this case A_γ is convex in the universal cover M and $A_\gamma/Z(\gamma)$ is compact. More generally it is known to be true for semi-hyperbolic groups [5].

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Sergei Ivanov, St. Petersburg department of Steklov Math Institute RAS, 191023, Fontanka 27, Saint Petersburg, Russia

E-mail address: `svivanov@pdmi.ras.ru`

Vitali Kapovitch, Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 2E4, Canada

E-mail address: `vtk@math.toronto.edu`